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LETTER TO THE EDITOR

Heisenberg subalgebra, dressing approach and super bi-Hamiltonian integrable system

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Abstract. Starting with the affine super-algebra $\widehat{Osp}(2,2)$, we have shown how different choices of Heisenberg subalgebras in conjunction with the dressing operator approach of Wilson leads to different super-symmetric bi-Hamiltonian systems. Subsequently, the explicit form of the symplectic operators are deduced with the help of the Trace identity. Lastly a possible relation with the τ-function is indicated.

Enlarging the class of integrable systems is one of the most important motivations of present-day research. Of late, several methodologies have been put forward, each with some distinct advantages and disadvantages. Of course the properties of the infinite dimensional Lie algebras [1] (loop algebras or Kac–Moody algebras) form the basis of all such approaches. In the study of the representation theory of such infinite dimensional Lie algebras, an essential role is always played by the Heisenberg subalgebra [2]. In this letter we have analysed the hierarchy of equations associated with the supersymmetric affine Lie algebra $\widehat{Osp}(2,2)$ [3] by using the various types of Heisenberg subalgebra of it. The method we follow is the dressing operator method of Wilson [4]. Subsequently we have deduced the bi-Hamiltonian form of these equations using the generalized form of the Trace identity [5]. We also indicate a possible relation with the τ-functions [6].

The supersymmetric Lie algebra $Osp(2,2)$ is generated by the following set of generators:

$$\{t_0, t_\pm, t_{(0)}, t_{\pm 1/2}, t_{(\pm 1/2)}\}$$

where t_0, $t_\pm$, $t_{(0)}$ are Bosonic and $t_{\pm 1/2}$, $t_{(\pm 1/2)}$ are Fermionic. A matrix belonging to the $Osp(2,2)$ Lie superalgebra will usually have the form:

$$\begin{pmatrix} B_1 & F_1 \\ F_2 & B_2 \end{pmatrix} \tag{1}$$

where B_1, B_2 are bosonic and F_1, F_2 are fermionic blocks. The matrix is supertraceless $\operatorname{tr} B_1 - \operatorname{tr} B_2 = 0$. The lowest dimensional representation of $Osp(2,2)$ is 3×3. The corresponding affine untwisted Lie superalgebra has the generators $t_0\lambda^m$, $t_\pm\lambda^m$, $t_{(0)}\lambda^m$, $t_{\pm 1/2}\lambda^m$, $t_{(\pm 1/2)}\lambda^m$ where m is an integer and $\lambda \in s^1$. The commutation rules are well known and we do not quote them here.

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In the Wilson's dressing approach, τ-function is written as the following vacuum expectation value

$$\tau_l(t) = \langle V_0, T^{-l}\Psi_{\mathrm{vac}(t)} n(x) V_0\rangle \tag{2}$$

where V_0 is the highest weight vector of the representation, $n(x)$ is a non-unique group element that produces the τ-function. For the definition of T and other details see [7]. Now

$$\Psi_{\mathrm{vac}}(t) = \exp\left(\sum_{i>0} p_i t_i\right) \tag{3}$$

where p_i are elements of the Heisenberg subalgebra, and t_i are complex variables in which we intend to study the evolution of the nonlinear fields.

An arbitrary group element n in the Kac–Moody group $Osp(2/2) \otimes \mathbb{C}[\lambda, \lambda^{-1}]$ can be factorized as follows:

$$\begin{aligned} n &= n_- n_0 n_+ \\ n_- &= 1 + \lambda^{-1} M_{-1} + \lambda^{-1/2} M_{-1/2} + \cdots \\ n_+ &= 1 + \lambda N_1 + \lambda^{1/2} N_{1/2} + \cdots \end{aligned} \tag{4}$$

where N_i, M_i are arbitrary elements of $Osp(2/2)$ with

$$n_0 = N_- \omega H N_+ \tag{5}$$

where $N_-(N_+)$ is the exponentiation of a strictly lower (upper) triangular matrix, ω is a Weyl group element and H is the exponentiation of an element of the Cartan subalgebra. Such a factorization is known as Birkhoff decomposition. Such a type of Birkhoff decomposition was used by Mulase and Ueno [8] for the supersymmetric systems.

Let us now set

$$\hat{\Psi}^l = T^{-l}\Psi_{\mathrm{vac}}(t) n(x). \tag{6}$$

Then by the Birkhoff decomposition of $\hat{\Psi}^l$ we can write

$$\hat{\Psi}^l = \hat{\Psi}^l_- \hat{\Psi}^l_0 \hat{\Psi}^l_+. \tag{7}$$

If now the ith evolution of $\hat{\Psi}^l$ obeys $\partial_{t_i}\hat{\Psi}^l = p_i\hat{\Psi}^l$, then from equation (7) we get

$$(\hat{\Psi}^l)^{-1}\partial_{t_i} \cdot \hat{\Psi}^l_- + (\partial_{t_i} \cdot \hat{\Psi}^l_{0,+})(\hat{\Psi}^l_{0,+})^{-1} = \hat{R}^l(p_i)$$

where $\hat{\Psi}^l_{0,+} = \hat{\Psi}^l_0 \hat{\Psi}^l_+$

$$\hat{R}^l_{(p_i)} = (\hat{\Psi}^l_-)^{-1} p_i \hat{\Psi}^l_-.$$

So if we write $\hat{R}^l(p_i) = \hat{R}^l_+(p_i) + \hat{R}^l_-(p_i)$ then we get two equations,

$$[\partial_{t_i} + \hat{R}^l_-(p_i)](\hat{\Psi}^l_-)^{-1} = 0 \tag{8}$$

$$[\partial_{t_i} - \hat{R}^l_+(p_i)]\hat{\Psi}^l_{0,+} = 0. \tag{9}$$

Equations (8) and (9) are the required Lax equations.

We now apply the above formalism to our case of $\widehat{Osp}(2,2)$. The elements of the Heisenberg subalgebra can be constructed either in a purely bosonic way or by keeping both bosonic and fermionic generators. For example,

$$\left.\begin{aligned} p_1 &= t_- + t_+\lambda \\ q_1 &= t_+ + t_-\lambda \end{aligned}\right\} \tag{10}$$

$$\left.\begin{aligned} p_2 &= t_+\lambda + t_- + (t_{(1/2)} - t_{(-1/2)})\theta\lambda^{1/2} \\ q_2 &= t_+ + t_-\lambda + (t_{(1/2)} - t_{(-1/2)})\theta\lambda^{-1/2} \end{aligned}\right\} \tag{11}$$

satisfy $[p_i, q_i] = k/2$ $(i = 1, 2)$.

In equation (11), θ stands for an anticommuting constant. To calculate $\hat{R}^l_+$, we start with p_2 and expand $\hat{\Psi}^l_-$ as:

$$\hat{\Psi}^l_- = 1 + \lambda^{-1}A_1 + \lambda^{-1/2}A_{-1/2} + \cdots$$
$$A_1 = \begin{pmatrix} \alpha_1 & \beta_1 & \mu_1 \\ \gamma_1 & \delta_1 & \nu_1 \\ \phi_1 & \psi_1 & \alpha_1 + \delta_1 \end{pmatrix} A_{-1/2} = \begin{pmatrix} a_1 & r_1 & s_1 \\ c_1 & b_1 & f_1 \\ g_1 & h_1 & a_1 + b_1 \end{pmatrix} \tag{12}$$

so that one gets

$$\hat{R}^l_+(p_2) = [(\hat{\Psi}^l_-)^{-1} p_2 (\hat{\Psi}^l_-)]_+ = t_+\lambda + t_- + [t_{+,A-1/2}]\lambda^{1/2} + (t_{(1/2)} - t_{(-1/2)})\theta\lambda^{1/2}$$
$$+ [t_{+,A-1}] + [(t_{(1/2)} - t_{(-1/2)})\theta, A_{-1/2}]. \tag{13}$$

With a redefinition of the unknown functions involved in the matrices A_1, $A_{-1/2}$, we can recast $\hat{R}^l_+(p_2)$ as:

$$\hat{R}^l_+(p_2) = \begin{pmatrix} -r_2 & \alpha_2 - \lambda^2 & c_2 + \theta\lambda \\ 1 & \beta_2 + r_2 & b_2 + \theta\lambda \\ 0 & \phi_2 & \beta_2 \end{pmatrix}. \tag{14}$$

To simplify, we choose $\beta_2 = r_2 = \beta$, $C_2 + \theta\lambda = b$, $b_2 + \theta\lambda = 0$ whence

$$\hat{R}^l_+(p_2) = \begin{pmatrix} -\beta & \alpha - \lambda^2 & b \\ 1 & 2\beta & 0 \\ 0 & \phi & \beta \end{pmatrix}. \tag{15}$$

Zero curvature equation can now be written as:

$$\frac{\partial J_x}{\partial t_i} = [J_i, J_x] + \frac{\partial J_i}{\partial x} \qquad i = 1, 2. \tag{16}$$

$J_x = \hat{R}^l_+(p_2)$, and $J_i \in Osp(2,2)$ is an arbitrary matrix written as

$$J_i = \begin{pmatrix} n & \gamma & \delta \\ c & m & f \\ g & h & n + m \end{pmatrix}. \tag{17}$$

So equation (16) immediately yields

$$
\begin{aligned}
\frac{\partial\beta}{\partial t_i} &= -bg + 2m' - 3c'\beta - 3c\beta' + c'' - \phi' f \\
\frac{\partial\alpha}{\partial t_i} &= 2C'(\alpha - \lambda^2) + 4bg\beta - 9m'\beta + 18C'\beta^2 + 18C\beta'\beta \\
&\qquad + 4\phi f\beta + c\alpha' + 2b'g + 3bg' - 3m'' + 12C'\beta' + 6c\beta'' - 2c''' + \phi' f \\
\frac{\partial\phi}{\partial t_i} &= g(\alpha - \lambda^2) + \beta(2g\beta - 2\phi c) + (m + c)\phi + g'\beta + 2g\beta' + \phi' c + \phi c' - g'' \\
\frac{\partial b}{\partial t_i} &= 2(cb - f\beta)\beta - (\alpha - \lambda^2) f - bm + c'b + cb' + f'\beta - f\beta' + f''.
\end{aligned}
\tag{18}
$$

In these equations we consider c, m, g, f as unknown functions and expand them as a power series in λ

$$
\begin{aligned}
c &= \sum_{j=0}^{n} C_j(\alpha, \beta, \phi, b)\lambda^{n-j} \\
m &= \sum_{j=0}^{n} E_j(\alpha, \beta, \phi, b)\lambda^{n-j} \\
g &= \sum_{j=0}^{n} B_j(\alpha, \beta, \phi, b)\lambda^{n-j} \\
f &= \sum_{j=0}^{n} F_j(\alpha, \beta, \phi, b)\lambda^{n-j}.
\end{aligned}
\tag{19}
$$

Substituting in (18) and equating powers of λ yields

$$
\begin{aligned}
\frac{\partial\beta}{\partial t_i} &= -bB_n + 2\partial E_n + (\partial^2 - 3\beta\partial - 3\beta')C_n - \phi F_n \\
\frac{\partial\alpha}{\partial t_i} &= (4b\beta + 2b' + 3b\partial)B_n - (9\beta\partial + 3\partial^2)E_n + (2\alpha\partial + 18\beta^2\partial + 18\beta'\beta + \alpha' \\
&\qquad + 12\beta'\partial + 6\beta'' - 2\partial^2)c_n + (4\phi\beta + \phi')E_n \\
\frac{\partial\phi}{\partial t_i} &= (\alpha + 2\beta^2 + \beta\partial + 2\beta' - \partial^2)\beta_n + \phi E_n + (-2\beta\phi + 2\phi\partial + \phi')C_n \\
\frac{\partial b}{\partial t_i} &= -bE_n + (2b\beta + b' + b\partial)C_n - (2\beta^2 + \alpha - \beta\partial + \beta' - \partial^2)F_n
\end{aligned}
\tag{20}
$$

giving the nth set of equations in the hierarchy.

We now consider symplectic structure. To identify the Poisson structure associated with the above nonlinear equation we observe that the Hamiltonian is given by

$$
H = K\left(J_1, \frac{\delta J_x}{\delta\lambda}\right) = \sum_k H_k \lambda^{-k}
\tag{21}
$$

where K is the Killing form given as $K(x, y) = \mathrm{str}(x, y)$, $x, y \in Osp(2, 2)$. The variational formalism for such zero curvature equation was given by Zhang, who showed that,

$$\left(\frac{\delta}{\delta U_j}\right) \mathrm{str}\left(\bar{J}_i \frac{\partial J_x}{\partial \lambda}\right) = \frac{\partial}{\partial \lambda} \mathrm{str}\left(\bar{J}_i \frac{\partial J_x}{\partial U_i}\right)$$

or

$$\left(\frac{\delta}{\delta U_i}\right)\left\langle \bar{J}_i, \frac{\partial J_x}{\partial \lambda}\right\rangle = \frac{\partial}{\partial \lambda}\left\langle \bar{J}_i, \frac{\partial J_x}{\partial U_i}\right\rangle \tag{22}$$

where U_i stands for the nonlinear field variables and $\bar{J} = \lambda^{\mu J}$, μ any positive integer. Equation (22) goes by the name of the Trace identity. Actually Zhang proved it for the case of ordinary Lie algebra which can easily be extended to the supersymmetric case by using graded products and graded commutators. In the following we will be using (22) to identify the coefficients occurring in (20) as the variational derivatives of H defined in (21). After explicit evaluation we get

$$\begin{aligned}
\frac{\delta H_k}{\delta \beta} &= 3(l-k)E_{n+k-l} \\
\frac{\delta H_k}{\delta \alpha} &= (l-k)C_{n+k-l} \\
\frac{\delta H_k}{\delta b} &= (l-k)B_{n+k-l} \qquad \text{and} \\
\frac{\delta H_k}{\delta \phi} &= (l-k)F_{n+k-l}
\end{aligned} \tag{23}$$

which at once permits us to identify the following Poisson-brackets forming the second symplectic structure:

$$\begin{aligned}
\{\beta(x), b(y)\} &= -b\delta(x-y) \\
\{\beta(x), \beta(y)\} &= 2\partial x \delta(x-y) \\
\{\beta(x), \alpha(y)\} &= (\partial^2 - 3\beta\partial - 3\beta')\delta(x-y) \\
\{\beta(x), \phi(y)\} &= -\phi\delta(x-y) \\
\{\alpha(x), b(y)\} &= (4b\beta + 2b' + 3b\partial)\delta(x-y) \\
\{\alpha(x), \alpha(y)\} &= (2\alpha\partial + 18\beta^2\partial + 18\beta'\beta + \alpha' + 12\beta'\partial + b\beta'' - 2\partial^2)\delta(x-y) \\
\{\alpha(x), \phi(y)\} &= (4\phi\beta + \phi')\delta(x-y) \\
\{\phi(x), b(y)\} &= (\alpha + 2\beta^2 + \beta\partial + 2\beta' - \partial^2)\delta(x-y) \\
\{\phi(x), \phi(y)\} &= \{b(x), b(y)\} = 0.
\end{aligned} \tag{24}$$

In our above analysis we have derived a new hierarchy of supersymmetric integrable systems by using the dressing operator approach of Wilson from a specific realization of the Heisenberg subalgebra. As noted previously there can be another choice for this subalgebra, generated by $p_1 \cdot q_1$ given in (10). For that case if we take

$$A_{-1} = \begin{pmatrix} \alpha_1 & \beta_1 & 0 \\ r_1 & \delta_1 & 0 \\ 0 & 0 & \alpha_1 + \delta_1 \end{pmatrix} \qquad A_{-1/2} = \begin{pmatrix} 0 & 0 & a_1 \\ 0 & 0 & b_1 \\ c_1 & d_1 & 0 \end{pmatrix}$$

then J_x turns out to be

$$J_x = \begin{pmatrix} \alpha & \beta - \lambda^2 & a\lambda \\ 1 & -\alpha & 0 \\ 0 & b\lambda & 0 \end{pmatrix} \tag{25}$$

and then we can repeat all the above calculations and arrive at a new hierarchy of equations. Finally, it is interesting to note that the four nonlinear fields occurring in equation (15) can be expressed as the following vacuum expectation values:

$$\begin{aligned}
\alpha &= -\langle \lambda^{-1} t_+ \cdot 1, \lambda^{-1} (\hat{\psi}^l_-)^{-1} \{t_+\lambda + t_- + (t_{(1/2)} - t_{(-1/2)})\theta\lambda^{1/2}\} \hat{\Psi}^l_- \cdot 1\rangle \\
b &= \langle \lambda^{-1} t_{(1/2)} \cdot 1, \lambda^{-1} (\hat{\psi}^l_-)^{-1} \{t_+\lambda + t_- + (t_{(1/2)} - t_{(-1/2)})\theta\lambda^{1/2}\} \hat{\Psi}^l_- \cdot 1\rangle \\
\phi &= -\langle \lambda^{-1} t_{1/2} \cdot 1, \lambda^{-1} (\hat{\psi}^l_-)^{-1} \{t_+\lambda + t_- + (t_{(1/2)} - t_{(-1/2)})\theta\lambda^{1/2}\} \hat{\psi}^l_- \cdot 1\rangle \\
\beta &= \langle \lambda^{-1} (3t_0 + 1/2 t_{(0)}) \cdot 1, \lambda^{-1} (\hat{\psi}^l_-)^{-1} \{t_+^\lambda + t_- + (t_{(1/2)} - t_{(-1/2)})\theta\lambda^{1/2}\} \hat{\psi}^l_- \cdot 1\rangle .
\end{aligned} \tag{26}$$

These expressions are obtained by the simple process of projection on the Heisenberg subalgebra. A glance at the Wilson definition of the τ-function (2) will convince one that it is possible to express these in terms of such τ-functions, but we do not pursue such studies here—we will report them in a future communication.

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